Post-Newtonian Approximation of the Vlasov-Nordström System

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Abstract

We study the Nordström-Vlasov system which describes the dynamics of a self-gravitating ensemble of collisionless particles in the framework of the Nordström scalar theory of gravitation. If the speed of light c is considered as a parameter, it is known that in the Newtonian limit $c \to \infty$ the Vlasov-Poisson system is obtained. In this paper we determine a higher approximation and establish a pointwise error estimate of order $\mathcal{O}(c^{-4})$. Such an approximation is usually called a 1.5 post-Newtonian approximation.

1 Introduction and Main Results

In astrophysics stars of a galaxy are often modeled by a collisionless gas interacting only by the gravitational fields which they create collectively; together this leads to the Vlasov-Einstein system. Recently, interest has arisen in a simplified but still relativistic model essentially going back to Nordström, see [19], in which the dynamic of the matter is coupled to a *scalar* theory of gravitation, the metric tensor potential is replaced by a scalar function and Einstein's equations are replaced by a wave equation. In [4] a reformulated version of Nordström's theory is presented in which the system reads

$$\left.\begin{aligned} S(f) - \Big[S(\phi)p + \gamma c^2 \nabla_x \phi\Big] \cdot \nabla_p f = 4S(\phi) f \\ \mu = \int \gamma f \, dp \\ -\partial_t^2 \phi + c^2 \Delta_x \phi = 4\pi\mu. \end{aligned}\right\} \tag{VNc}$$

In the previous equations, $f = f(t, x, p)$ gives the probability density to find a particle at time t at position x with momentum p where $t \in \mathbb{R}$, $x \in \mathbb{R}^3$, $p \in \mathbb{R}^3$. $\phi = \phi(t, x)$ is the mean Nordström gravitational potential generated by the particles, c the speed of light. Moreover

$$p^2 = |p|^2, \quad \gamma = (1 + c^2 p^2)^{-1/2}, \quad \hat{p} = \gamma p, \quad \text{and} \quad S = \partial_t + \hat{p} \cdot \nabla_x$$

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is the relativistic free-streaming operator and $\hat{p}$ the relativistic velocity associated to the momentum p.

Together with the initial conditions

$$f(0,x,p) = f^{\circ}(x,p), \quad \phi(0,x) = \phi^0(x), \quad \partial_t\phi(0,x) = \phi^1(x) \tag{1.1}$$

(VNc) is the Cauchy problem of the Vlasov-Nordström system. For a physical interpretation and a derivation of this system see [3].

In this formulation (VNc) exhibits many similarities to the relativistic Vlasov-Maxwell system, which also consists of a, in that case homogenous, Vlasov equation coupled to a linear hyperbolic equation. Thus, many techniques developed for the Vlasov-Maxwell system also apply for the Vlasov-Nordström system providing existence and uniqueness of local classical solutions as well as a continuation criterion, which guarantees that the solution is global in time if a control for the velocity support of the matter density is established, see [4] . Furthermore, there are results about existence of global weak solutions, see [5], existence of global classical solutions in the 2D case, see [15], for spherically symmetric initial data, see [1], and for small initial data, see [6]. Concerning the Vlasov-Maxwell system we refer the reader to [7] and the references cited therein. On the other hand the same important pieces of the basic theory are missing, namely global existence of classical solutions with unrestricted initial data and uniqueness of weak solutions. However, it is remarkable that there is a blow-up result for (VNc) if the sign in the force-term of the Vlasov equation is changed, see [4].

This paper is concerned with the non-relativistic limit of (VNc), i.e. the limit $c \to \infty$: From physical intuition the Vlasov-Poisson system is the non-relativistic limit of the Vlasov-Einstein system, a statement which has been made rigorous under certain circumstances in [22]. In [3] it has been shown that as $c \to \infty$ also solutions of (VNc) converge to a solution of a Vlasov-Poisson equation in a pointwise sense obeying an error estimate of order $\mathcal{O}(c^{-1})$. Thus, (VNc) can be regarded as another relativistic generalization of the Vlasov-Poisson system. The situation in the Maxwell case is similar as solutions of the Vlasov-Maxwell system approach a solution of the Vlasov-Poisson system (in the plasma case, which differs from the Vlasov-Poisson system for the gravitational case by the sign in the force-term) in the same sense, see [23].

It is the goal of this paper to replace the Vlasov-Poisson system, the classical or Newtonian limit, by another effective equation to achieve higher order convergence and a more precise approximation (in the plasma case this has been done in [2]). This will lead to an effective system whose solution stay as close as $\mathcal{O}(c^{-4})$ to a solution of (VNc) if the initial data are matched appropriately. Thus, it is the 1.5 Post-Newtonian approximation, note that in the context of General Relativity the approximations are usually counted in powers of c^{-2}.

In order to derive effective equations in the limit of small initial velocities we shall formally expand all quantities in powers of c^{-1}

$$\begin{aligned}
f &= f_0 + c^{-1} f_1 + c^{-2} f_2 + \cdots \\
\mu &= \mu_0 + c^{-1} \mu_1 + c^{-2} \mu_2 + \cdots \\
\phi &= \phi_0 + c^{-1} \phi_1 + c^{-2} \phi_2 + \cdots
\end{aligned}$$

put this ansatz into (VNc) and derive an equation in every order of c^{-1}.

Therefore, we obtain in the orders c^2 and c^1

$$-\Delta_x \phi_0 = 0, \quad -\Delta_x \phi_1 = 0,$$

thus, we set

$$\phi_0 = \phi_1 = 0. \tag{1.2}$$

In the zeroth order we obtain a Vlasov-Poisson System

$$\left.\begin{array}{ll} \tilde{S}(f_0) - \nabla_x \phi_2 \cdot \nabla_p f_0 = 0, & \mu_0 = \int f_0 \, dp, \\ \phi_2(t,x) = -\int |z|^{-1} \mu_0(t, x+z) \, dz, & f_0(0,x,p) = f^\circ(x,p) \end{array}\right\} \tag{VP}$$

where

$$\tilde{S} = \partial_t + p \cdot \nabla_x$$

is the non-relativistic free-streaming operator. In the first order a linearized Vlasov-Poisson system appears

$$\left.\begin{array}{c} \tilde{S}(f_1) - \nabla_x \phi_3 \cdot \nabla_p f_0 - \nabla_x \phi_2 \cdot \nabla_p f_1 = 0, \\ \mu_1 = \int f_1 \, dp, \qquad \Delta_x \phi_3 = \mu_1 + \partial_t^2 \phi_1. \end{array}\right\}$$

Hence, if we suppose that $f_1(0,x,p) = 0$, we can set

$$f_1 = 0 \quad \text{and} \quad \phi_3 = 0,$$

which also yields

$$\mu_1 = 0. \tag{1.3}$$

In the second order we derive an inhomogeneous Vlasov equation coupled to a Poisson equation

$$\left.\begin{array}{c} \tilde{S}(f_2) - \nabla_x \phi_2 \cdot \nabla_p f_2 - \nabla_x \phi_4 \cdot \nabla_p f_0 = \\ 4 f_0 \tilde{S}(\phi_2) + \frac{p^2}{2} p \cdot \nabla_x f_0 + \left(\tilde{S}(\phi_2) p - \frac{p^2}{2} \nabla_x \phi_2\right) \cdot \nabla_p f_0 \\ \mu_2 = \int (f_2 - 1/2 p^2 f_0) \, dp, \qquad \Delta_x \phi_4 = \mu_2 + \partial_t^2 \phi_2 \end{array}\right\} \tag{LVP}$$

for which we choose homogeneous initial data

$$f_2(0,x,p) = 0. \tag{1.4}$$

At this point we need to discuss the Poisson equation for ϕ_4. Since $\Delta_x \partial_t^2 \phi_2 = 4\pi \partial_t^2 \mu_0$ the Poisson equation in (LVP) can be rewritten as

$$\phi_4 = \Delta_x^{-1}(4\pi \mu_2) + (\Delta_x)^{-2}(4\pi \partial_t^2 \mu_0). \tag{1.5}$$

Therefore, we define ϕ_4 by

$$\phi_4(t,x) = -\int\int |z|^{-1} \big(f_2 - 1/2 p^2 f_0\big)(t, x+z, p) \, dp \, dz - \frac{1}{2} \int |z| \partial_t^2 f_0(t, x+z, p) \, dp \, dz. \tag{1.6}$$

Using (VP) and partial integration the second term can be rewritten as

$$-\frac{1}{2} \int |z| \partial_t^2 f_0(t, x+z, p) \, dp \, dz = -\frac{1}{2} \int\int (\bar{z} \cdot p) \partial_t f_0(t, x+z, p) \, dp \, dz, \qquad \bar{z} = z|z|^{-1}.$$

Thus far this term is merely bounded instead of decaying at infinity and ϕ_4 seems to be determined only modulo a function depending on t. But if the boundary condition is fixed in terms of

integrability in weighted Sobolev spaces (for a definition see section 2, below (2.3)) instead of a pointwise estimates, the corresponding condition for vanishing at infinity is

$$\phi_4 \in L^p_s \qquad \text{for every } s < -2 + 3/p' \tag{1.7}$$

where $1 < p < \infty,\ 1/p + 1/p' = 1$. We will show in section 2 that in fact ϕ_4 fulfills this integrability condition.

The first aim of this paper is to show that

$$\begin{aligned} f^D &:= f_0 + c^{-2} f_2, \\ \phi^D &= c^{-2}\phi_2 + c^{-4}\phi_4 \end{aligned} \tag{1.8}$$

yields a higher order pointwise approximation of the Vlasov-Nordström than the Vlasov-Poisson system. We will call this system the Darwin system because in the case of individual charged particles the relevant approximation is usually called the Darwin approximation. It is clear that for achieving this improved approximation property also the initial data of the Vlasov-Nordström model has to be matched appropriately by the data for the Darwin system. For a prescribed initial density f° we are able to calculate (f_0, ϕ_2) and (f_2, ϕ_4) according to what has been outlined above. We then consider the Vlasov-Nordström system with the initial condition

$$\begin{aligned} f(0,x,p) &= f^\circ(x,p) \\ \phi^0(x) = \phi(0,x) &= c^{-2}\phi_2(0,x) + c^{-4}\phi_4(0,x) \\ \phi^1(x) = \partial_t\phi(0,x) &= c^{-2}\partial_t\phi_2(0,x) + c^{-4}\partial_t\phi_4(0,x). \end{aligned} \tag{1.9}$$

Before we formulate the main theorem of this paper let us recall that solutions of the Vlasov-Nordström system together with the initial conditions (1.9) exist at least on some time interval $[0,T]$ which is independent of $c \geq 1$; see [3, Thm. 3]. This time interval is fixed throughout this paper.

Theorem 1.1 *Assume that $f^\circ \in C^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ is nonnegative and has compact support. From f° calculate (f_0, ϕ_2) and (f_2, ϕ_4), and then define initial data for the Vlasov-Nordström system by (1.9). Let (f, ϕ) denote the solution of the Vlasov-Nordström system with initial data (1.9) and let (f^D, ϕ^D) be defined as in (1.8). Then there exists a constant $M > 0$, and also for every $R > 0$ there is $M_R > 0$, such that*

$$\begin{aligned} |f(t,x,p) - f^D(t,x,p)| &\leq Mc^{-4} \quad && (x \in \mathbb{R}^3), \\ |\phi(t,x) - \phi^D(t,x)| &\leq M_R\, c^{-4} && (|x| \leq R), \\ |\partial_t\phi(t,x) - \partial_t\phi^D(t,x)| &\leq Mc^{-4} && (x \in \mathbb{R}^3), \\ |\nabla_x\phi(t,x) - \nabla_x\phi^D(t,x)| &\leq M_R\, c^{-6} && (|x| \leq R) \end{aligned} \tag{1.10}$$

for all $p \in \mathbb{R}^3$, $t \in [0,T]$ and $c \geq 1$.

The constants M and M_R are independent of $c \geq 1$ but do depend on the initial data. Note that if the Vlasov-Nordström system is compared to the Vlasov-Poisson system only, one obtains the estimates $|f(t,x,p) - f_0(t,x,p)| + c^2|\phi(t,x) - \phi_2(t,x)| + c^2|\nabla_x\phi(t,x) - \nabla_x\phi_2(t,x)| + |\partial_t\phi(t,x)| \leq Mc^{-1}$; see [3, Thm. 3].

While this approximation has the big advantage that, since by now the Vlasov-Poisson system is well understood, the existence of (f_0, ϕ_2) and also of (f_2, ϕ_4) does not pose serious problems;

note that in (LVP) the equation for f_2 is linear. Therefore one can hope to get more information on (VNc) by studying the approximate equations.

As a drawback of the above hierarchy one has to deal with two densities $f_0,\ f_2$ and two fields ϕ_2 and ϕ_4 to define f^D. Thus it is natural to look for a model which can be written down using only one density and one field. The most natural candidate for such a system might be the following.

$$\left.\begin{aligned} \partial_t f + p\Big(1-\frac{p^2}{2c^2}\Big)\cdot\nabla_x f - \Big[\tilde{S}(\phi)p + c^2\Big(1-\frac{p^2}{2c^2}\Big)\nabla_x\phi\Big]\cdot\nabla_p f = 4\tilde{S}(\phi) f \\ \mu = \int\Big(1-\frac{p^2}{2c^2}\Big) f(\cdot,\cdot,p)\,dp \\ \phi = \frac{4\pi}{c^2}\Delta^{-1}\mu + \frac{4\pi}{c^4}\Delta^{-2}\partial_t^2\mu \end{aligned}\right\}$$

Note, that (f^D,ϕ^D) solves this system up to an error of order c^{-4}. But since second derivatives occur it is not clear which initial conditions are to be posed. It turns out that it is more convenient to rewrite this system in terms of derivatives of ϕ then in terms of ϕ. To this end we introduce the scalar force field ψ^* corresponding to $\partial_t\phi$ and the vector field E^* corresponding to $\nabla\phi$. This leads to the following system which we call Darwin Vlasov-Nordström system.

$$\left.\begin{aligned} \partial_t f^* + p\Big(1-\frac{p^2}{c^2}\Big)\cdot\nabla_x f^* - \Big[(\psi^*+p\cdot E^*)p + c^2\Big(1-\frac{p^2}{2c^2}\Big)E^*\Big]\cdot\nabla_p f^* \\ = 4(\psi^*+p\cdot E^*)f^* \\ \mu^* = \int\Big(1-\frac{p^2}{2c^2}\Big) f^*(\cdot,\cdot,p)\,dp, \qquad j^* = \int p f^*(\cdot,\cdot,p)\,dp \\ \Delta\psi^* = -\frac{4\pi}{c^2}\nabla\cdot j^*, \qquad c^2\Delta E^* = 4\pi\nabla\mu^* + \partial_t\nabla\psi^* \end{aligned}\right\} \qquad \text{(DVN}\mathsf{c}\text{)}$$

Again $(f^D,\partial_t\phi^D,\nabla_x\phi^D)$ solves (DVNc) up to an error of order c^{-4}.

Theorem 1.2 *Assume that $f^\circ\in C^\infty(\mathbb{R}^3\times\mathbb{R}^3)$ is nonnegative and has compact support. Then there exist $c^*\geq 1$ and $T^*>0$ such that the following holds for $c\geq c^*$.*

(a) If there is a local solution of (DVNc), then the initial data ψ° and E° of (DVNc) at $t=0$ are uniquely determined by the initial density f°.

(b) The system (DVNc) has a unique C^2-solution (f^*,ψ^*,E^*) on $[0,T^*]$ attaining that initial data $(f^\circ,\psi^\circ,E^\circ)$ at $t=0$.

(c) Define ϕ^* by

$$\phi^* = \frac{4\pi}{c^2}\Delta^{-1}\mu^* - \frac{4\pi}{c^4}\Delta^{-2}\big(\nabla\cdot\partial_t j^*\big). \tag{1.11}$$

Then $c^2\Delta\phi^* = 4\pi\mu^* + \partial_t\psi^*$ and $\nabla_x\phi^* = E^*$ as well as $\partial_t\phi^* = \psi^* + \mathcal{O}(c^{-4})$.

Let (f,ϕ) denote the solution of (VNc) with initial data $(f^\circ,\phi^*(0,\cdot),\partial_t\phi^*(0,\cdot))$. Then there exists a constant $M>0$, and also for every $R>0$ there is $M_R>0$, such that

$$\begin{aligned} |f(t,x,p)-f^*(t,x,p)| &\leq Mc^{-4} \quad (x\in\mathbb{R}^3),\\ |\nabla_x\phi(t,x)-E^*(t,x)| &\leq M_R\,c^{-6} \quad (|x|\leq R),\\ |\partial_t\phi(t,x)-\psi^*(x,t)| &\leq Mc^{-4} \quad (x\in\mathbb{R}^3),\\ |\phi(t,x)-\phi^*(t,x)| &\leq M\,c^{-4} \quad (x\in\mathbb{R}^3), \end{aligned}$$

for all $p\in\mathbb{R}^3$, $t\in[0,\min\{T,T^*\}]$, and $c\geq c^*$.

We shall continue with a few remarks about the Darwin approximation of the relativistic Vlasov-Maxwell system derived in [2]. In that paper only the 1 PN approximation is treated but we want to emphasize that in the case of only one species it is possible to establish the 1.5 PN approximation for Vlasov-Maxwell as well in the same way as it is done for (VN$\mathtt{c}$) in this paper. On the other hand, if several species with different charge to mass ratio, say f^+ with charge 1 and f^- with charge -1, are under consideration, mass is set to unity for both species, an effect due to radiation damping takes place, which is proportional to

$$c^{-3}\partial_t \int\int \nabla U_0(f_0^+ + f_0^-)(t,x,p)\,dp\,dx \tag{1.12}$$

where $(f_0^\pm, U_0)$ is a solution of the Vlasov-Poisson system in the plasma case with two different species

$$\left.\begin{aligned} \tilde{S}(f^\pm) \pm \nabla_x U_0 \cdot \nabla_p f^\pm = 0 \\ \Delta U_0 = 4\pi \int (f_0^+ - f_0^-)\,dp. \end{aligned}\right\} \tag{VP$\mathtt{plasma}$}$$

(In the case of only one species and in the gravitational case the corresponding term vanishes, see (3.8) below.) Note that $\partial_t \int\int \nabla U_0(f_0^+ + f_0^-)(t,x,p)\,dp\,dx = \frac{d^3}{dt^3}D(t)$ where

$$D(t) := \int\int x(f_0^+ - f_0^-)(t,x,p)\,dp\,dx \tag{1.13}$$

is the dipole moment. Thus it seems reasonable to replace (VP$\mathtt{plasma}$) by

$$\left.\begin{aligned} \tilde{S}(f^\pm) \pm \Big(\frac{a}{c^3}\frac{d^3}{dt^3}D(t) + \nabla_x U_0\Big) \cdot \nabla_p f^\pm = 0 \\ \Delta U_0 = 4\pi \int (f_0^+ - f_0^-)\,dp. \\ D(t) = \int\int x(f_0^+ - f_0^-)(t,x,p)\,dp\,dx \end{aligned}\right\} \tag{VP$\mathtt{rad}$}$$

where $a \neq 0$ is a constant which has to be calculated from the full system. However, in this system phase space has increased by 3 dimensions and it is not so clear how to single out the initial data of $\frac{d^2}{dt^2}D$, note that $D(0)$ and $\frac{d}{dt}D(0)$ are already determined by the initial data of $f^\pm$. On a formal level (VP$\mathtt{rad}$) can be reduced in several ways, which are discussed in [10] and [11].

In the gravitational case effects due to damping are expected to take place in the 2.5 PN approximation and to be connected with the fifth time derivative of the quadrupole moment, see [11]. Thus, it seems reasonable that approximations up to the order of 2PN conserve an energy which can be calculated by an expansion of the energy

$$\mathcal{E} = c^2 \int\int \sqrt{1 + c^{-2}p^2} f(t,x,p)\,dp\,dx + c^2 \int \Big(|\partial_t \phi(t,x)|^2 + c^2|\nabla_x \phi(t,x)|^2\Big)\,dx, \tag{1.14}$$

which is conserved by solutions of the full Nordström system; note that in the Vlasov-Maxwell case the corresponding statement is true for approximations up to the 1PN level. However, here this fails already in the Newtonian approximation in a rather obscure manner; the conserved energy of (VP) is $\int\int \frac{p^2}{2} f\,dp\,dx - \int |\nabla_x \phi_2|^2\,dx$ whereas from the expansion of (1.14) one would expect $\int\int \frac{p^2}{2} f\,dp\,dx + \int |\nabla_x \phi_2|^2\,dx$ to be conserved. But this is the energy of Vlasov-Poisson system in the plasma case.

The paper is organized as follows. Some facts concerning (VP), (LVP), and (VNc) are collected in Section 2 as well as the proof of (1.7). The proof of Theorem 1.1 is elaborated in Section 3. Section 4 contains the proof of Theorem 1.2. For these proofs we will mostly rely on suitable representation formulas for the fields (refined versions of those used in [4, 3]), which are derived in the appendix, Section 5.

Notation: $B(0,R)$ denotes the closed ball in $\mathbb{R}^3$ with center at $x=0$ or $p=0$ and radius $R>0$. The usual L^∞-norm of a function $\phi=\phi(x)$ over $x\in\mathbb{R}^3$ is written as $\|\phi\|_x$, and if $\phi=\phi(x,p)$, we modify this to $\|\phi\|_{x,p}$. For $m\in\mathbb{N}$ the $W^{m,\infty}$-norms are denoted by $\|\phi\|_{m,x}$, etc. If $T>0$ is fixed, then we write

$$g(t,x,p,c)=\mathcal{O}_{cpt}(c^{-m}),$$

if for all $R>0$ there is a constant $M=M_R>0$ such that

$$|g(t,x,p,c)|\le Mc^{-m} \tag{1.15}$$

for $|x|\le R$, $p\in\mathbb{R}^3$, $t\in[0,T]$, and $c\ge 1$. Similarly, we write

$$g(t,x,p,c)=\mathcal{O}(c^{-m})$$

if there is a constant $M>0$ such that (1.15) holds for all $x,p\in\mathbb{R}^3$, $t\in[0,T]$ and $c\ge 1$. In general, generic constants are denoted by M.

2 Some properties of (VP), (LVP), and (VNc)

There is a vast literature on (VP), see e.g. [7, Sect. 4] or [21] and the references therein. For our purposes we collect a few well known facts about classical solutions of (VP).

Proposition 2.1 *Assume that $f^\circ\in C^\infty(\mathbb{R}^3\times\mathbb{R}^3)$ is nonnegative and has compact support. Then there exists a unique global C^1-solution f_0 of (VP), and there are nondecreasing continuous functions $P_{VP},K_{VP}:[0,\infty[\to\mathbb{R}$ such that*

$$\begin{gathered}\|f_0(t)\|_{x,p}\le\|f^\circ\|_{x,p},\\ \operatorname{supp} f_0(t,\cdot,\cdot)\subset B(0,P_{VP}(t))\times B(0,P_{VP}(t)),\\ \|f_0(t)\|_{1,x,p}+\|\phi_2(t)\|_{2,x}\le K_{VP}(t),\end{gathered} \tag{2.1}$$

for $t\in[0,\infty[$.

This result was first established by Pfaffelmoser [20], and simplified versions of the proof were obtained by Schaeffer [24] and Horst [9]; a proof along different lines is due to Lions and Perthame [17].

For our approximation scheme we also need bounds on higher derivatives of the solution. This point was elaborated in [16] where it was shown that if $f^\circ\in C^k(\mathbb{R}^3\times\mathbb{R}^3)$, then f_0 possesses continuous partial derivatives w.r.t. x and p up to order k. The existence of continuous time-derivatives then follows from the Vlasov equation. Thus, f_0 and ϕ_2 are C^∞ if f° is C^∞, and by a redefinition of K_{VP} we can assume that

$$\|f_0(t)\|_{5,x,p}\le K_{VP}(t),\quad t\in[0,\infty[. \tag{2.2}$$

The existence of a unique C^1-solution f_2 of (LVP) follows by a contraction argument, but we omit the details. Furthermore it can be shown that there are nondecreasing continuous functions $P_{LVP}, K_{LVP} : [0,\infty[\to \mathbb{R}$ such that

$$\operatorname{supp} f_2(\cdot,\cdot,t) \subset B(0,P_{LVP}(t)) \times B(0,P_{LVP}(t)), \tag{2.3}$$
$$\|f_2(t)\|_{2,x,p} + \|\phi_4\|_{2,x} \leq K_{LVP}(t), \tag{2.4}$$

for $t \in [0,\infty[$.

Next we shall investigate the boundary behavior of ϕ_4 in terms of integrability in weighted Sobolev spaces where the weight function is defined by $\rho(x) = (1+|x|^2)^{1/2}$. For $k \in \mathbb{N}_0$, $1 \leq p < \infty$ and $s \in \mathbb{R}$ we say $u \in W_s^{k,p}$ if and only if $u \in W_{\mathrm{loc}}^{k,p}(\mathbb{R}^3)$ and $\rho^{s+|\alpha|}\partial^\alpha u \in L^p(\mathbb{R}^3)$ for $0 \leq |\alpha| \leq k$; $\alpha \in \mathbb{N}_0^3$ a usual multi-index. In order to establish (1.7) we use some mapping properties of the Laplacian in weighted Sobolev spaces due to McOwen. Citing [18], for $1 < p < \infty$, $1/p + 1/p' = 1$, we have that $\phi_4 \in W_s^{2,p}$ for $s < -2 + 3/p'$ if $\partial_t^2\phi_2 \in L_{2+s}^p$. Furthermore, $\partial_t^2\phi_2 \in L_{2+s}^p$ holds true if

$$\Delta\partial_t^2\phi_2 = \partial_t^2\mu_0 \in L_{4+s}^p \qquad \text{and} \qquad \int h(x)\partial_t^2\mu_0(x)\,dx = 0 \tag{2.5}$$

for every polynomial h of degree less than 1. Thus, to prove (1.7) it is sufficient to show (2.5). Since $\partial_t^2\mu_0(t,\cdot)$ has compact support for every $t \geq 0$, compare (2.1), the first condition is clear. Now let $h(x) = a_0 + a_1x_1 + a_2x_2 + a_3x_3$, $a_i \in \mathbb{R}$. Employing (VP), some partial integrations yield

$$\begin{aligned}
&\int h(x)\partial_t^2\mu_0(t,x)\,dx = -\int\int h(x)p\cdot\nabla_x\partial_t f_0(t,x,p)\,dp\,dx \\
&= \nabla h \cdot \int\int p\partial_t f_0(t,x,p)\,dp\,dx = \nabla h \cdot \int\int p(-p\cdot\nabla_x f_0 + \nabla_x\phi_2\cdot\nabla_p f_0)(t,x,p)\,dp\,dx \\
&= -\nabla h \cdot \int \nabla\phi_2\mu_0(t,x)\,dx = -\frac{1}{4\pi}\sum_{i,j=1}^{3} a_i \int \partial_i\phi_2\partial_j\partial_j\phi_2(t,x)\,dx \\
&= \frac{1}{8\pi}\nabla h \cdot \int \nabla|\nabla\phi_2(x)|^2\,dx = 0.
\end{aligned}$$

Concerning solutions of (VNc) we have the following from [3, Thm. 3, Proposition 2].

Proposition 2.2 *Assume that $f^\circ \in C^\infty(\mathbb{R}^3\times\mathbb{R}^3)$ is nonnegative and has compact support. If ϕ^0 and ϕ^1 are defined by (1.9), then there exits $T > 0$ (independent of c) such that for all $c \geq 1$ the system (VNc) with initial data (1.9) has a unique C^1-solution (f,ϕ) on the time interval $[0,T]$. In addition, there are nondecreasing continuous functions (independent of c) $P_{VN}, K_{VN} : [0,T] \to \mathbb{R}$ such that*

$$f(x,p,t) = 0 \quad \text{if} \quad |p| \geq P_{VN}(t), \tag{2.6}$$
$$|f(t)|_{x,p} + |\phi(t)|_{1,x} \leq K_{VN}(t), \tag{2.7}$$

for all $x \in \mathbb{R}^3$, $t \in [0,T]$, and $c \geq 1$.

3 Proof of Theorem 1.1

The proof follows the usual lines developed in [23] and in [2] for higher order approximations. In a first step the difference of the fields, $\phi - \phi^D$ and the difference of its derivatives is estimated in

terms of the difference of the densities $h = f - f^D$. For that purpose we shall use quite elaborated representations of the fields, which are derived in the appendix. Next we will calculate which Vlasov equation it is that h fulfills. Using the matching of the initial data and a Gronwall argument, it follows that $|h| = \mathcal{O}(c^{-4})$ which in turn yields the announced estimates of the errors in the fields.

As $\nabla\phi$ enters the Vlasov equation with the factor c^2 we have to be most precise in comparing $\nabla\phi$ with $\nabla\phi^D$. Therefore, we give the reasoning for that term in some detail. In section 5.1.1 below we will show that the gradient of the approximate field ϕ^D from (1.8) admits the following representation.

$$\nabla_x\phi^D = \phi^D_{x,\mathtt{ext}} + \phi^D_{x,\mathtt{int}} + \phi^D_{x,\mathtt{bd}} + \mathcal{O}_{cpt}(c^{-6}) \tag{3.1}$$

with

$$\begin{aligned}
\phi^D_{x,\mathtt{ext}}(t,x) &= -\frac{1}{c^2}\int_{|z|\geq ct}\int |z|^{-2}\bar{z}\Big(f^D - \frac{p^2}{2c^2}f_0\Big)(t,x+z,p)\,dp\,dz \\
&\quad +\frac{1}{2c^4}\int_{|z|\geq ct}\int \bar{z}\partial_t^2 f_0(t,x+z,p)\,dp\,dz, \qquad (3.2)\\
\phi^D_{x,\mathtt{int}}(t,x) &= -\frac{1}{c^2}\int_{|z|\leq ct}\int |z|^{-2}\bar{z}f^D(\hat{t}(z),x+z)\,dz \\
&\quad +\frac{1}{c^3}\int_{|z|\leq ct}\int |z|^{-2}\big(2\bar{z}(\bar{z}\cdot p) - p\big)f^D(\hat{t}(z),x+z,p)\,dp\,dz \\
&\quad +\frac{1}{c^4}\int_{|z|\leq ct}\int |z|^{-2}\big(-3\bar{z}(\bar{z}\cdot p)^2 + (\bar{z}\cdot p)p + \frac{3}{2}\bar{z}p^2\big)f^D(\hat{t}(z),x+z,p)\,dp\,dz \\
&\quad -\frac{1}{c^4}\int_{|z|\leq ct}\int |z|^{-1}\bar{z}\bar{z}\cdot\nabla_x\phi_2 f^D(\hat{t}(z),x+z,p)\,dp\,dz \\
&\quad +\frac{1}{c^5}\int_{|z|\leq ct}\int |z|^{-2}\Big(4(\bar{z}\cdot p)^3\bar{z} - (\bar{z}\cdot p)^2 p - 4(\bar{z}\cdot p)p^2\bar{z} + p^2 p\Big)f^D(\hat{t}(z),x+z,p)\,dp\,dz \\
&\quad +\frac{1}{c^5}\int_{|z|\leq ct}\int |z|^{-1}\bar{z}\Big(2(\bar{z}\cdot p)\bar{z}\cdot\nabla_x\phi_2 - p\cdot\nabla_x\phi_2 - \tilde{S}(\phi_2)\Big)f^D(\hat{t}(z),x+z,p)\,dp\,dz \\
&\quad -\frac{1}{3c^5}\int_{|z|\leq ct}\partial_t\Big(\nabla_x\phi_2\mu_0\Big)(\hat{t}(z),x+z)\,dz, \qquad (3.3)\\
\phi^D_{x,\mathtt{bd}}(t,x) &= \frac{1}{c^4 t}\int_{|z|=ct}\int \bar{z}(\bar{z}\cdot p)\Big(1 - \frac{(\bar{z}\cdot p)}{c} + \frac{(\bar{z}\cdot p)^2 - p^2}{c^2}\Big)f^\circ(x+z,p)\,dp\,ds(z) \\
&\quad -\frac{t}{3c^4}\int_{|z|=ct}\int \bar{z}(\bar{z}\cdot p)\partial_t^2 f_0(0,x+z,p)\,dp\,ds(z) \\
&\quad -\frac{1}{3c^5}\int_{|z|=ct}\int (\bar{z}\cdot p)p\partial_t f_0(0,x+z,p)\,dp\,ds(z)
\end{aligned}$$

where the subscripts '`ext`', '`int`' and '`bd`' refer to the exterior, interior and boundary integration in z. We also recall that $\bar{z} = |z|^{-1}z$ and $\hat{t}(z) = t - c^{-1}|z|$ is the retarded time. On the other hand, according to Section 5.1.2 below, we have

$$\nabla_x\phi = \phi_{x,\mathtt{ext}} + \phi_{x,\mathtt{int}} + \phi_{x,\mathtt{bd}} + \mathcal{O}_{cpt}(c^{-6}) \tag{3.4}$$

with

$$\phi_{x,\mathtt{ext}}(t,x) = -\frac{1}{c^2}\int_{|z|\geq ct}\int |z|^{-2}\bar{z}\Big(f_0 + t\partial_t f_0 + \frac{t^2}{2}\partial_t^2 f_0 + \frac{t^3}{6}\partial_t^3 f_0\Big)(0,x+z,p)\,dp\,dz$$

$$
\begin{aligned}
& +\frac{1}{2c^4}\int_{|z|\geq ct}\int |z|^{-2}\bar{z}p^2(f_0+t\partial_t f_0)(0,x+z,p)\,dp\,dz \\
& +\frac{1}{2c^4}\int_{|z|\geq ct}\int \bar{z}\Big(\partial_t^2 f_0+t\partial_t^3 f_0\Big)(0,x+z,p)\,dp\,dz \\
& -\frac{t}{c^4}\int_{|z|\geq ct}\int |z|^{-2}\bar{z}\partial_t f_2(0,x+z,p)\,dp\,dz, \qquad (3.5)
\end{aligned}
$$

$$
\begin{aligned}
\phi_{x,\mathtt{int}}(t,x) \;=\; & -\frac{1}{c^2}\int_{|z|\leq ct}\int |z|^{-2}\bar{z}f(\hat{t}(z),x+z)\,dz \\
& +\frac{1}{c^3}\int_{|z|\leq ct}\int |z|^{-2}\big(2\bar{z}(\bar{z}\cdot p)-p\big)f(\hat{t}(z),x+z,p)\,dp\,dz \\
& +\frac{1}{c^4}\int_{|z|\leq ct}\int |z|^{-2}\big(-3\bar{z}(\bar{z}\cdot p)^2+(\bar{z}\cdot p)p+\frac{3}{2}\bar{z}p^2\big)f(\hat{t}(z),x+z,p)\,dp\,dz \\
& -\frac{1}{c^2}\int_{|z|\leq ct}\int |z|^{-1}\bar{z}\bar{z}\cdot\nabla_x\phi f(\hat{t}(z),x+z,p)\,dp\,dz \\
& +\frac{1}{c^5}\int_{|z|\leq ct}\int |z|^{-2}\Big(4(\bar{z}\cdot p)^3\bar{z}-(\bar{z}\cdot p)^2p-4(\bar{z}\cdot p)p^2\bar{z}+p^2p\Big)f_0(\hat{t}(z),x+z,p)\,dp\,dz \\
& +\frac{1}{c^3}\int_{|z|\leq ct}\int |z|^{-1}\bar{z}\Big(2(\bar{z}\cdot p)\bar{z}\cdot\nabla_x\phi-p\cdot\nabla_x\phi-\tilde{S}(\phi)\Big)f(\hat{t}(z),x+z,p)\,dp\,dz \\
& -\frac{1}{3c^5}\int_{|z|\leq ct}\partial_t\Big(\nabla_x\phi_2\mu_0\Big)(0,x+z)\,dz, \qquad (3.6) \\
\phi_{x,\mathtt{bd}}(t,x) \;=\; & \phi^D_{x,\mathtt{bd}}(t,x).
\end{aligned}
$$

In order to verify (1.10) we start by comparing the exterior fields. Let $x\in B(0,R)$ with $R>0$ be fixed. Then we obtain from (3.2) and (3.5), taking into account that

$$
\int f_2(0,x,p)\,dp=0
$$

by (LVP) and (1.4) as well as (2.1), (2.2), (2.3) and (2.4),

$$
\begin{aligned}
& |\nabla_x\phi_{\mathtt{ext}}(t,x)-\nabla_x\phi^D_{\mathtt{ext}}(t,x)| \\
\leq\; & \frac{1}{c^2}\int_{|z|\geq ct}|z|^{-2}\Big|\mu_0(t,x+z)-\big(\mu_0+t\partial_t\mu_0+\frac{t^2}{2}\partial_t^2\mu_0+\frac{t^3}{6}\partial_t^3\mu_0\big)(0,x+z)\Big|\,dz \\
& +\frac{1}{2c^4}\int_{|z|\geq ct}\Big|\partial_t^2\mu_0(t,x+z)-\big(\partial_t^2\mu_0+t\partial_t^3\mu_0\big)(0,x+z)\Big|\,dz \\
& +\frac{1}{c^4}\int_{|z|\geq ct}\int |z|^{-2}|f_2(t,x+z,p)-\big(f_2+t\partial_t f_2\big)(0,x+z,p)|\,dp\,dz \\
& +\frac{1}{2c^4}\int_{|z|\geq ct}\int |z|^{-2}p^2|f_0(t,x+z,p)-\big(f_0+t\partial_t f_0\big)(0,x+z,p)|\,dp\,dz \\
\leq\; & \frac{M}{c^2}\int_{|z|\geq ct}|z|^{-2}\bigg(\int_0^t (t-s)^3P_{VP}(s)^3K_{VP}(s)\mathbf{1}_{B(0,P_{VP}(s))}(x+z)\,ds\bigg)\,dz \\
& +\frac{M}{c^4}\int_{|z|\geq ct}\bigg(\int_0^t (t-s)P_{VP}(s)^3K_{VP}(s)\mathbf{1}_{B(0,P_{VP}(s))}(x+z)\,ds\bigg)\,dz
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{M}{c^4}\int_{|z|\geq ct}|z|^{-2}\Big(\int_0^t(t-s)P_{VP}(s)^5K_{VP}(s)\mathbf{1}_{B(0,P_{VP}(s))}(x+z)\,ds\Big)\,dz\\
&+\frac{M}{c^4}\int_{|z|\geq ct}|z|^{-2}\int_0^t(t-s)\Big(P_{LVP}(s)^3K_{LVP}(s)\mathbf{1}_{B(0,P_{LVP}(s))}(x+z)\,ds\Big)\,dz\\
\leq\ &\frac{Mt^4}{c^2}\int_{|z|\geq ct}|z|^{-2}\mathbf{1}_{B(0,R+M_0)}(z)\,dz+\frac{Mt^2}{c^4}\int_{|z|\geq ct}\big(|z|^{-2}+1\big)\mathbf{1}_{B(0,R+M_0)}(z)\,dz\\
\leq\ &M_Rc^{-6};
\end{aligned}
\tag{3.7}
$$

note that here we have used

$$
M_0=\max_{s\in[0,T]}\Big(P_{VP}(s)+K_{VP}(s)+P_{LVP}(s)+K_{LVP}(s)\Big)<\infty
$$

and for instance

$$
\frac{t^4}{c^2}\int_{|z|\geq ct}|z|^{-2}\mathbf{1}_{B(0,R+M_0)}(x+z)\,dz\leq\frac{t^4}{c^6t^4}\int_{|z|\leq R+M_0}|z|^2\,dz\leq M_Rc^{-6}.
$$

To bound $|\phi_{x,\mathtt{int}}(t,x)-\phi^D_{x,\mathtt{int}}(t,x)|$ we shall first treat the last term in (3.3) and (3.6) respectively, which is special in some way as it does not arise from the expansion of the integral kernels of $\nabla_x\phi$. Regarding this term, a crucial observation is that by (VP) and partial integration

$$
\begin{aligned}
\int\nabla_x\phi_2\mu_0(t,x)\,dx&=\frac{1}{4\pi}\Big(\sum_{j=1}^3\int\partial_{x_i}\phi_2\partial_{x_j}\partial_{x_j}\phi_2(t,x)\,dx\Big)_{i=1,2,3}\\
&=-\frac{1}{4\pi}\Big(\sum_{j=1}^3\int\partial_{x_i}\partial_{x_j}\phi_2\partial_{x_j}\phi_2(t,x)\,dx\Big)_{i=1,2,3}=-\frac{1}{8\pi}\int\nabla_x|\nabla_x\phi_2(t,x)|^2\,dx=0
\end{aligned}
\tag{3.8}
$$

for all $t\in[0,\infty[$. This yields

$$
\begin{aligned}
-\frac{1}{3c^5}\int_{|z|\leq ct}\partial_t\Big(\mu_0\nabla_x\phi_2\Big)(\hat t(z),x+z)\,dz&=-\frac{1}{3c^5}\int_{|z|\leq ct}\partial_t\Big(\mu_0\nabla_x\phi_2\Big)(t,x+z)\,dz+\mathcal{O}_{cpt}(c^{-6})\\
=\ &-\frac{\partial_t}{3c^5}\Big(\int_{|z|\leq ct}\Big(\mu_0\nabla_x\phi_2\Big)(t,x+z)\,dz\Big)+\frac{1}{3c^4}\int_{|z|=ct}\Big(\mu_0\nabla_x\phi_2\Big)(0,x+z)\,ds(z)+\mathcal{O}_{cpt}(c^{-6})\\
=\ &\frac{\partial_t}{3c^5}\Big(\int_{|z|\geq ct}\Big(\mu_0\nabla_x\phi_2\Big)(t,x+z)\,dz\Big)+\frac{1}{3c^4}\int_{|z|=ct}\Big(\mu_0\nabla_x\phi_2\Big)(0,x+z)\,ds(z)+\mathcal{O}_{cpt}(c^{-6})\\
=\ &\frac{1}{3c^5}\int_{|z|\geq ct}\partial_t\Big(\mu_0\nabla_x\phi_2\Big)(t,x+z)\,dz+\mathcal{O}_{cpt}(c^{-6})
\end{aligned}
\tag{3.9}
$$

where we used that for $x\in B(0,R)$

$$
\begin{aligned}
&\Big|\int_{|z|\leq ct}\partial_t\Big(\mu_0\nabla_x\phi_2\Big)(\hat t(z),x+z)-\partial_t\Big(\mu_0\nabla_x\phi_2\Big)(t,x+z)\,dz\Big|\\
&\qquad\leq\ \frac{M}{c}\int_{|z|\leq ct}|z|K_{VP}^2(t)P_{VP}^3(t)\mathbf{1}_{B(0,P_{VP})}(x+z)\,dz\\
&\qquad\leq\ \frac{M}{c}\int|z|\mathbf{1}_{R+M_0}(z)\,dz\leq M_Rc^{-1}
\end{aligned}
$$

(note that $|\hat{t}(z)| \leq t$ for $|z| \leq ct$ and K_{VP} and P_{VP} are nondecreasing). Again by (3.8) and same calculations we also have

$$-\frac{1}{3c^5}\int_{|z|\leq ct} \partial_t\Big(\mu_0\nabla_x\phi_2\Big)(0,x+z)\,dz = \frac{1}{3c^5}\int_{|z|\geq ct} \partial_t\Big(\mu_0\nabla_x\phi_2\Big)(0,x+z)\,dz. \tag{3.10}$$

Employing (3.9) and (3.10) we can estimate

$$\begin{aligned}
&\frac{1}{3c^5}\bigg|\int_{|z|\leq ct} \partial_t\Big(\mu_0\nabla_x\phi_2\Big)(\hat{t}(z),x+z) - \partial_t\Big(\mu_0\nabla_x\phi_2\Big)(0,x+z)\,dz\bigg| \\
&\qquad \leq \frac{M}{c^5}\bigg|\int_{|z|\geq ct} \partial_t\Big(\mu_0\nabla_x\phi_2\Big)(t,x+z) - \partial_t\Big(\mu_0\nabla_x\phi_2\Big)(0,x+z)\,dz\bigg| + M_R c^{-6} \\
&\qquad \leq \frac{M}{c^6}\int_{|z|\geq ct} |z| K_{VP}^2(t) P_{VP}(t)^3 \mathbf{1}_{B(0,P_{VP}(t))}(x+z)\,dz + M_R c^{-6} \leq M_R c^{-6} \qquad (3.11)
\end{aligned}$$

for all $t \in [0,T]$ and $x \in B(0,R)$.

Next we recall from [3, Thm.3] that

$$|\nabla_x\phi(t,x) - c^{-2}\nabla_x\phi_2(t,x)| = \mathcal{O}(c^{-3}); \tag{3.12}$$

actually in [3] the initial conditions are different, but we only added terms of order c^{-4} so that an inspection of the proof in [3] leads to (3.12). Furthermore, in section 5.1.1, (5.5) and section 5.1.2, (5.31) it is shown that

$$|\partial_t\phi^D(t,x)| + |\partial_t\phi(t,x)| = \mathcal{O}(c^{-2}). \tag{3.13}$$

Also, we define

$$H(t) = \sup\{|f(s,x,p) - f_0(s,x,p) - c^{-2}f_2(s,x,p)| \,:\, x\in\mathbb{R}^3,\ p\in\mathbb{R}^3,\ s\in[0,t]\}. \tag{3.14}$$

Now we have to proceed in two steps: Using (3.12) and (3.13) prove

$$\begin{aligned}
|\nabla_x\phi^D(t,x) - \nabla_x\phi(t,x)| &\leq \frac{1}{c^2}M_R(c^{-3} + H(t)) \\
|\partial_t\phi^D(t,x) - \partial_t\phi(t,x)| &\leq M(c^{-3} + H(t)),
\end{aligned}$$

employ this to derive

$$H(t) \leq Mc^{-3}$$

for all $t \in [0,T]$, which in turn gives

$$\begin{aligned}
|\nabla_x\phi^D(t,x) - \nabla_x\phi(t,x)| &\leq M_R c^{-5} \\
|\partial_t\phi^D(t,x) - \partial_t\phi(t,x)| &\leq Mc^{-3}
\end{aligned}$$

and especially as $c^{-4}\nabla_x\phi_2 = \mathcal{O}(c^{-4})$

$$|c^{-2}\nabla_x\phi_0(t,x) - \nabla_x\phi(t,x)| \leq M_R c^{-4}. \tag{3.15}$$

In the second step this procedure will be repeated where estimate (3.12) is replaced by (3.15), which will give the announced estimates. To avoid redundancies we shall only carry out the second step, and therefore assume that (3.15) is already proved.

Introducing the constant

$$M_1 = \max_{s\in[0,T]} \Big(P_{VN}(s) + P_{VP}(s) + P_{LVP}(s) \Big) < \infty$$

it follows that $f(s,x,p) = f_0(s,x,p) = f_2(s,x,p) = 0$ for $x \in \mathbb{R}^3$, $|p| > M_1$ and $s \in [0,T]$. Also if $R_0 > 0$ is chosen such that $f^\circ(x,p) = 0$ for $|x| \geq R_0$, defining the constant

$$M_2 = R_0 + TM_1 + \max_{s\in[0,T]} \Big(P_{VP}(s) + P_{LVP}(s) \Big) < \infty, \tag{3.16}$$

we have that $f(s,x,p) = f_0(s,x,p) = f_2(s,x,p) = 0$ for $|x| \geq M_2$, $p \in \mathbb{R}^3$ and $s \in [0,T]$. Let again $x \in B(0,R)$ with $R > 0$ be fixed. From (3.6), (3.3), (3.11), (3.8), (3.12), (3.15) (1.9) and $0 \leq \hat{t}(z) \leq t$ for $|z| \leq ct$ we obtain

$$\begin{aligned}
&|\phi_{\mathtt{int}}(t,x) - \phi^D_{\mathtt{int}}(t,x)| \\
&\leq \frac{1}{c^2} \int_{|z|\leq ct} \int |z|^{-2} \Big| (f - f^D)(\hat{t}(z), x+z, p) \Big| \, dp \, dz \\
&\quad + \frac{1}{c^3} \int_{|z|\leq ct} \int |z|^{-2} \big| 2\bar{z}(\bar{z}\cdot p) - p \big| \Big| (f - f^D)(\hat{t}(z), x+z, p) \Big| \, dp \, dz \\
&\quad + \frac{1}{c^4} \int_{|z|\leq ct} \int |z|^{-2} \Big| -3\bar{z}(\bar{z}\cdot p)^2 + (\bar{z}\cdot p)p + \frac{3}{2}\bar{z}p^2 \Big| \Big| (f - f^D)(\hat{t}(z), x+z, p) \Big| \, dp \, dz \\
&\quad + \frac{1}{c^2} \int_{|z|\leq ct} \int |z|^{-1} \Big| (\nabla_x \phi - \frac{1}{c^2} \nabla_x \phi_0 \Big| f(\hat{t}(z), x+z, p) \, dp \, dz \\
&\quad + \frac{1}{c^4} \int_{|z|\leq ct} \int |z|^{-1} \Big| \nabla_x \phi_2 (f - f^D)(\hat{t}(z), x+z, p)) \Big| \, dp \, dz \\
&\quad + \frac{1}{c^5} \int_{|z|\leq ct} \int |z|^{-2} \Big| 4(\bar{z}\cdot p)^3 \bar{z} + (\bar{z}\cdot p)^2 p - 4(\bar{z}\cdot p)p^2\bar{z} + \frac{3p^2}{2} p \Big| \Big| (f - f^D)(\hat{t}(z), x+z, p) \Big| \, dp \, dz \\
&\quad + \frac{M}{c^3} \int_{|z|\leq ct} \int |z|^{-1} \Big(|p| \Big| \nabla_x \phi - \frac{1}{c^2} \nabla_x \phi_2 \Big| + \Big| \tilde{S}(\phi) - \frac{1}{c^2} \tilde{S}(\phi_2) \Big| \Big) f(\hat{t}(z), x+z, p) \, dp \, dz \\
&\quad + \frac{M}{c^5} \int_{|z|\leq ct} \int |z|^{-1} \Big(|p| |\nabla_x \phi_2| + |\tilde{S}(\phi_2)| \Big) \Big| f - f^D \Big| (\hat{t}(z), x+z, p) \, dp \, dz + M_R c^{-6} \\
&\leq \frac{M}{c^2} (M_1^3 + M_1^6) H(t) \int_{|z|\leq ct} |z|^{-2} \mathbf{1}_{B(0,M_2)}(x+z) \, dz \\
&\quad + \frac{M}{c^4} (M_1^3 + M_1^4) K_{VP}(T) H(t) \int_{|z|\leq ct} |z|^{-1} \mathbf{1}_{B(0,M_2)}(x+z) \, dz \\
&\quad + \frac{M_R}{c^6} (M_1^4 + M_1^4) ||f^\circ||_{x,p} \int_{|z|\leq ct} \mathbf{1}_{B(0,M_2)}(x+z) \, dz + \frac{M_R}{c^6} \\
&\leq c^{-2} M_R (H(t) + c^{-4})
\end{aligned} \tag{3.17}$$

since for instance

$$\int_{|z|\leq ct} |z|^{-1} \mathbf{1}_{B(0,M_2)}(x+z) \, dz \leq \int_{|z|\leq R+M_2} |z|^{-1} \, dz \leq M_R.$$

Recalling that $\phi_{x,\mathtt{bd}} = \phi^D_{x,\mathtt{bd}}$, we can summarize (3.1), (3.4), (3.7) and (3.17) as

$$c^2 |\nabla_x \phi(t,x) - \nabla_x \phi^D(t,x)| \leq M_R (H(t) + c^{-4}) \tag{3.18}$$

for $x \in B(0, R)$ and $t \in [0, T]$. Formulas (5.4),(5.28), (5.5), (5.30) and an analog calculation also lead to

$$\begin{aligned} |\phi(t,x) - \phi^D(t,x)| &\le M(H(t) + c^{-4}) && (3.19)\\ |\partial_t\phi(t,x) - \partial_t\phi^D(t,x)| &\le M(H(t) + c^{-4}) && (3.20)\end{aligned}$$

for $t \in [0, T]$. It remains to estimate $h = f - f^D$. Using (VNc), (1.8), (VP) and (LVP), it is found that

$$\begin{aligned} & S(h) - \Big[S(\phi)p + \gamma c^2\nabla_x\phi\Big] \cdot \nabla_p h \\ = \ & -S(f^D) + \Big[S(\phi)p + \gamma c^2\nabla_x\phi\Big] \cdot \nabla_p f^D + 4S(\phi)f \\ = \ & -\Big[\hat{p} - p + \frac{1}{2c^2}p^2p\Big] \cdot \nabla_x f_0 - \frac{1}{c^2}(\hat{p} - p) \cdot \nabla_x f_2 + \Big[S(\phi) - \frac{1}{c^2}\tilde{S}(\phi_2)\Big]p \cdot \nabla_p f_0 \\ & + \frac{1}{c^2}S(\phi)p \cdot \nabla_p f_2 + \Big[\gamma c^2\nabla_x\phi - \Big(1 - \frac{p^2}{2c^2}\Big)\nabla_x\phi_2 - \frac{1}{c^2}\nabla_x\phi_4\Big] \cdot \nabla_p f_0 \\ & + \Big[\gamma\nabla_x\phi - \frac{1}{c^2}\nabla_x\phi_2\Big] \cdot \nabla_p f_2 + 4S(\phi)h - 4\Big(\frac{1}{c^2}\tilde{S}(\phi_2) - S(\phi)\Big)f_0 + \frac{4}{c^2}S(\phi)f_2. \end{aligned}$$

If $|p| \le M_1$, then also $|\hat{p}| = (1 + c^{-2}p^2)^{-1/2}|p| \le |p| \le M_1$ uniformly in c, and hence

$$\frac{1}{c^2}\Big|\hat{p} - p\Big| + \Big|\hat{p} - p + \frac{1}{2c^2}p^2p\Big| \le Mc^{-4}.$$

In view of the bounds (2.1), (2.4) and (2.7), $S(\phi) = \mathcal{O}(c^{-2})$ as shown in section 5.1.2, (5.23) and (5.31), thus by (3.18) and (3.20)

$$\begin{aligned} & \Big|S(h)(t,x,p) - \Big[S(\phi)(t,x,p) + \gamma c^2\nabla_x\phi(t,x)\Big] \cdot \nabla_p f(t,x,p)\Big| \\ & \qquad \le M\Big(c^{-4} + H(t)\Big) \qquad (3.21)\end{aligned}$$

for all $|x| \le M_2$, $|p| \le M_1$ and $t \in [0, T]$. But in $\{(t, x, p) : |x| > M_2\} \cup \{(t, x, p) : |p| > M_1\}$ we have $h = f - f^D = 0$ by the above definition of M_1 and M_2. Accordingly, (3.21) is satisfied for all $x \in \mathbb{R}^3$, $p \in \mathbb{R}^3$ and $t \in [0, T]$. Since $h(0, x, p) = 0$ the argument from [3] (see also [23, p.416]) yields

$$H(t) \le \int_0^t M(c^{-4} + H(s))ds,$$

and therefore $H(t) \le Mc^{-4}$ for $t \in [0, T]$. Then, due to (3.19)–(3.20) $c^2|\nabla_x\phi(t, x) - \nabla_x\phi^D(t, x)| \le M_R c^{-4}$ for $|x| \le R$ and $t \in [0, T]$ as well as $|\phi(t, x) - \phi^D(t, x)| + |\partial_t\phi(t, x) - \partial_t\phi^D(t, x)| \le Mc^{-4}$ for $x \in \mathbb{R}^3$ and $t \in [0, T]$. This completes the proof of Theorem 1.1. □

4 Proof of Theorem 1.2

In this section we will be sketchy and omit many details, since the proof is more or less a repetition of what has been said before. First let us assume that there is a C^2-solution (f^*, ψ^*, E^*) of (DVNc) existing on a time interval $[0, T^*]$ for some $T^* > 0$ such that $\operatorname{supp} f^*(t, \cdot, \cdot) \subset \mathbb{R}^3 \times \mathbb{R}^3$ is compact for all $t \in [0, T^*]$. Then, after partial integration

$$\psi^*(t,x) = \frac{1}{c^2}\int\int |z|^{-2}\bar{z} \cdot pf^*(t, x + p, p)\, dp\, dz \qquad (4.1)$$

$$\Delta E^*(t,x) \;=\; \frac{4\pi}{c^2}\big(\nabla\mu^* + \partial_t\nabla\psi^*\big). \tag{4.2}$$

Since $f^*(0,x,p) = f^\circ(x,p)$, it follows that $\psi^*(0,\cdot)$ is determined by f°.

In order to compute the Poisson integral for E^*, we calculate by means of the transformation $y = w - z,\ dy = dw$, and using (5.35) below,

$$\begin{aligned}
&\frac{1}{c^2}\Big(\Delta^{-1}\partial_t\nabla\psi^*\Big)(t,x)\\
&= -\frac{1}{c^2 4\pi}\int |x-y|^{-1}\nabla\partial_t\psi^*(t,y)\,dy\\
&= -\frac{1}{c^4 4\pi}\int |x-y|^{-1}\nabla_y\Big(\int\int |z|^{-2}(\bar z\cdot p)\partial_t f^*(t,y+z,p)\,dp\,dz\Big)\,dy\\
&= -\frac{1}{c^4 4\pi}\Big[\int\Big(\int |z|^{-2}\bar z|x-w-z|^{-1}\,dz\Big)\cdot\int p\partial_{w_i}\partial_t f^*(t,w,p)\,dp\,dw\Big]_{i=1,2,3}\\
&= -\frac{1}{2c^4}\int\int |w-x|^{-1}(w-x)\cdot p\nabla_w\partial_t f^*(t,w,p)\,dp\,dw\\
&= -\frac{1}{2c^4}\int\int (\bar z\cdot p)\nabla_z\partial_t f^*(t,x+z,p)\,dp\,dz\\
&= \frac{1}{2c^4}\int\int |z|^{-1}\big(-\bar z(\bar z\cdot p)+p\big)\partial_t f^*(t,x+z,p)\,dp\,dz.
\end{aligned}$$

If we invoke the Vlasov equation for f^* and integrate by parts, this can be rewritten as

$$\begin{aligned}
&\frac{1}{c^2}\Big(\Delta^{-1}\partial_t\nabla\psi^*\Big)(t,x)\\
&= \frac{1}{2c^4}\int\int |z|^{-1}\Big(-\bar z(\bar z\cdot p)+p\Big)\Big(-(1-\frac{p^2}{2c^2})p\cdot\nabla_x f^*\\
&\quad +\Big[(\psi^*+p\cdot E^*)p + c^2\Big(1-\frac{p^2}{2c^2}\Big)E^*\Big]\cdot\nabla_p f^* + 4(\psi^*+p\cdot E^*)f^*\Big)(t,x+z,p)\,dp\,dz\\
&= \frac{1}{2c^4}\int\int |z|^{-2}\Big(-2(\bar z\cdot p)p + 3(\bar z\cdot p)^2\bar z - p^2\bar z\Big)\Big(1-\frac{p^2}{2c^2}\Big)f^*(t,x+z,p)\,dp\,dz\\
&\quad -\frac{1}{2c^4}\int\int |z|^{-1}\Big\{(1-\bar z\otimes\bar z)\Big[(\psi^*+p\cdot E^*)p + c^2\Big(1-\frac{p^2}{2c^2}\Big)E^*\Big]f^*\\
&\qquad\qquad +\big(\bar z(\bar z\cdot p)-p\big)\big(\psi^*+p\cdot E^*\big)f^*\Big\}(t,x+z,p)\,dp\,dz.
\end{aligned}$$

Therefore the solution E^* of (4.2) has the representation

$$\begin{aligned}
E^*(t,x) &= \Delta^{-1}\Big(\frac{4\pi}{c^2}\nabla\mu^*\Big) + \Delta^{-1}\Big(\frac{1}{c^2}\nabla\partial_t\psi^*\Big)\\
&= -\frac{1}{c^2}\int |z|^{-2}\bar z\mu^*(t,x+z)\,dz\\
&\quad +\frac{1}{2c^4}\int\int |z|^{-2}\Big(-2(\bar z\cdot p)p + 3(\bar z\cdot p)^2\bar z - p^2\bar z\Big)\Big(1-\frac{p^2}{2c^2}\Big)f^*(t,x+z,p)\,dp\,dz\\
&\quad -\frac{1}{2c^4}\int\int |z|^{-1}\Big\{(1-\bar z\otimes\bar z)\Big[(\psi^*+p\cdot E^*)p + c^2\Big(1-\frac{p^2}{2c^2}\Big)E^*\Big]f^*
\end{aligned}$$

$$+\big(\bar z(\bar z\cdot p)-p\big)\big(\psi^*+p\cdot E^*\big)f^*\Big\}(t,x+z,p)\,dp\,dz. \tag{4.3}$$

In particular, if we evaluate this relation at $t=0$ the Banach fix point theorem applied in $C_b(\mathbb{R}^3)$ shows that for $c\geq c^*$ sufficiently large the function $E^*(0,\cdot)$ is uniquely determined by $f^\circ = f^*(0,\cdot,\cdot)$. Thus f° alone fixes ψ° and E°.

Concerning the local and uniform (in c) existence of a solution to (DVNc) one can use (4.1) and (4.3) to follow the usual method by setting up an iteration scheme for which convergence can be verified on a small time interval: cf. [7, Sect. 5.8] and also [4]. For comparison of E^* with $\nabla_x\phi^D$ we give an alternative expression of $\nabla_x\phi^D$. Using (VP) and partial integration twice we have

$$\begin{aligned}
&-\nabla_x\frac{1}{2c^4}\int|z|\partial_t^2\mu_0(t,x+z)\,dz=\frac{1}{2c^4}\int\bar z\partial_t^2\mu_0(t,x+z)\,dz\\
&\quad=\frac{1}{2c^4}\int\int\bar z\partial_t\Big(-p\cdot\nabla_x f_0+\nabla_x\phi_2\cdot\nabla_p f_0\Big)(t,x+z,p)\,dp\,dz\\
&\quad=\frac{1}{2c^4}\int\int|z|^{-1}\Big(-(\bar z\cdot p)\bar z+p\Big)\partial_t f_0(t,x+z,p)\,dp\,dz\\
&\quad=\frac{1}{2c^4}\int\int|z|^{-1}\Big(-(\bar z\cdot p)\bar z+p\Big)\Big(-p\cdot\nabla_x f_0+\nabla_x\phi_2\cdot\nabla_p f_0\Big)(t,x+z,p)\,dp\,dz\\
&\quad=\frac{1}{2c^4}\int\int|z|^{-2}\Big(-2(\bar z\cdot p)p+3(\bar z\cdot p)^2\bar z-p^2\bar z\Big)f_0(t,x+z,p)\,dp\,dz\\
&\qquad-\frac{1}{2c^4}\int|z|^{-1}(1-\bar z\otimes\bar z)\nabla_x\phi_2\mu_0(t,x+z)\,dz.
\end{aligned}\tag{4.4}$$

Together (5.1), (4.4) (4.3) and (5.5), (4.1) reveals the analogy of E^* to $\nabla_x\phi^D$ and ψ^* to $\partial_t\phi^D$ respectively at the relevant orders of c^{-1}. Comparison of (1.11) with (VP), (LVP) and (1.8) gives the analogy of the initial values of (VNc). Finally, by similar arguments as used in the proof of Theorem 1.1 it can be shown that solutions of (DVNc) approximate solutions of (VNc) up to an error of order c^{-4}.

5 Appendix

5.1 Representation Formulas

5.1.1 Representation of the approximation force field

Here we will present representation formulas for the approximate field and its derivatives. As the calculations are lengthy and arguments are similar we shall only sketch the computations leading to the formula for $\nabla_x\phi^D$ and merely give the formulas for ϕ^D and $\partial_t\phi^D$.

We recall $\phi^D=c^{-2}\phi_2+c^{-4}\phi_4$ where

$$\begin{aligned}
\phi_2(t,x)&=-\int|z|^{-1}\mu_0(t,x+z)\,dz\\
\phi_4(t,x)&=-\frac12\int|z|\partial_t^2\mu_0(t,x+z)\,dz-\int|z|^{-1}\mu_2(t,x+z)\,dz,
\end{aligned}$$

thus, using $\mu_0=\int f_0(\cdot,\cdot,p)\,dp$ and $\mu_2=\int\Big(f_2-\frac{p^2}{2}f_0\Big)(\cdot,\cdot,p)\,dp$

$$\nabla_x\phi^D(t,x)=-\frac{1}{c^2}\int\int|z|^{-2}\bar z\Big(f^D-\frac{p^2}{2c^2}f_0\Big)(t,x+z,p)\,dp\,dz$$

$$+\frac{1}{2c^4}\int \bar{z}\partial_t^2\mu_0(t,x+z)\,dz. \tag{5.1}$$

We split the domain of integration in $\{|z|>ct|\}$ and $\{|z|<ct\}$; note that the exterior part gives $\phi_{x,\mathtt{ext}}$. To handle the interior part $\{|z|<ct\}$ we expand the densities w.r.t. t about the retarded time

$$\hat{t}(z) := t - c^{-1}|z|.$$

To begin with we have

$$\begin{aligned}
&-\frac{1}{c^2}\int_{|z|\le ct}|z|^{-2}\bar{z}\mu_0(t,x+z)\,dz\\
&= \; -\frac{1}{c^2}\int_{|z|\le ct}|z|^{-2}\bar{z}\Big(1+\frac{|z|}{c}\partial_t+\frac{|z|^2}{2c^2}\partial_t^2+\frac{|z|^3}{6c^3}\partial_t^3\Big)\mu_0(\hat{t}(z),x+z)\,dz\\
&\quad -\frac{1}{6c^2}\int_{|z|\le ct}|z|^{-2}\bar{z}\int_{\hat{t}(z)}^t (t-s)^3\partial_t^4\mu_0(s,x+z)\,ds\,dz
\end{aligned} \tag{5.2}$$

Since $\partial_t\mu_0+\int p\cdot\nabla_x f_0\,dp=0$ by (VP), we also find

$$\begin{aligned}
-\frac{1}{c^3}\int_{|z|\le ct}|z|^{-1}\bar{z}\partial_t\mu_0(\hat{t}(z),x+z)\,dz &= \frac{1}{c^3}\int_{|z|\le ct}|z|^{-1}\bar{z}\Big(\int p\cdot\nabla_x f_0(\hat{t}(z),x+z,p)\,dp\,dz\\
= \; & \frac{1}{c^3}\int_{|z|\le ct}\int |z|^{-1}\bar{z}p\cdot\Big(\nabla_z[f_0(\hat{t}(z),x+z,p)]+c^{-1}\bar{z}\,\partial_t f_0(\hat{t}(z),x+z,p)\Big)\,dp\,dz\\
= \; & \frac{1}{c^4 t}\int_{|z|=ct}\int \bar{z}(\bar{z}\cdot p)f^\circ(x+z,p)\,dp\,ds(z)\\
& -\frac{1}{c^3}\int_{|z|\le ct}\int |z|^{-2}\Big(-2\bar{z}(\bar{z}\cdot p)+p\Big)f_0(\hat{t}(z),x+z,p)\,dp\,dz\\
& +\frac{1}{c^4}\int_{|z|\le ct}\int |z|^{-1}\bar{z}(\bar{z}\cdot p)\,\partial_t f_0(\hat{t}(z),x+z,p)\,dp\,dz,
\end{aligned} \tag{5.3}$$

observe that $\hat{t}(z)=0$ for $|z|=ct$ was used for the boundary term. Using (2.1), (2.2) and (2.4) the last term in (5.3) as well as the remaining terms coming from the interior part of (5.1) and from (5.2) are in $\mathcal{O}_{cpt}(c^{-4})$; note that $|x|<R$ for some $R>0$ together with the support properties of f_0 imply that we only have to integrate in z over a set which is uniformly bounded in $c\ge 1$. Thus the leading terms in the orders c^{-2} and c^{-3} are specified. The following scheme emerges,

- expand the several terms in the interior part of (5.1) up to order c^{-5}, the error term is in $\mathcal{O}_{cpt}(c^{-6})$;
- replace time derivatives on the densities f_0 and f_2 via Vlasov's equations by space derivatives according to (VP) and (LVP);
- carry out partial integrations, using

  $$(\nabla_x g)(\hat{t}(z),x+z)=\nabla_z[g(\hat{t}(z),x+z)]+c^{-1}\bar{z}\partial_t g(\hat{t}(z),x+z)$$

  in the case of derivatives with respect to x.

Following this scheme one can calculate one after another the leading terms up to order c^{-5}.

Addition of some additional terms in $\mathcal{O}(c^{-6})$, as e.g. $\frac{1}{2c^6}\int_{|z|\le ct}\int |z|^{-2}\bar{z}p^2 f_2(\hat{t}(z), x+z, p)\,dp\,dz$, then gives (3.3) and (3.4).

One can also calculate

$$\phi^D = \phi^D_{\mathtt{ext}} + \phi^D_{\mathtt{int}} + \phi^D_{\mathtt{bd}} + \mathcal{O}(c^{-4}) \tag{5.4}$$

with

$$\begin{aligned}
\phi^D_{\mathtt{ext}}(t,x) &= -\frac{1}{c^2}\int_{|z|>ct} |z|^{-1}\big(\mu_0 + c^{-2}\mu_2\big)(x+z,t)\,dz \\
&\quad -\frac{1}{2c^4}\int_{|z|\ge ct} |z|\partial_t^2\mu_0(t,x+z)\,dz, \\
\phi^D_{\mathtt{int}}(t,x) &= -\frac{1}{c^2}\int_{|z|\le ct}\int |z|^{-1} f^D(\hat{t}(z), x+z, p)\,dp\,dz, \\
\phi^D_{\mathtt{bd}}(t,x) &= +\frac{1}{c^3}\int_{|z|=ct}\int (\bar{z}\cdot p)\, f^\circ(x+z,p)\,dp\,ds(z)
\end{aligned}$$

and

$$\partial_t\phi^D = \phi^D_{t,\mathtt{ext}} + \phi^D_{t,\mathtt{int}} + \phi^D_{t,\mathtt{bd}} + \mathcal{O}(c^{-4}) \tag{5.5}$$

is derived in the same manner with

$$\begin{aligned}
\phi^D_{t,\mathtt{ext}}(t,x) &= \frac{1}{c^2}\int_{|z|\ge ct}\int |z|^{-2}(\bar{z}\cdot p) f^D(t, x+z, p)\,dp\,dz, \\
\phi^D_{t,\mathtt{int}}(t,x) &= \frac{1}{c^2}\int_{|z|\le ct}\int |z|^{-2}(\bar{z}\cdot p) f^D(\hat{t}(z), x+z, p)\,dp\,dz \\
&\quad -\frac{1}{c^3}\int_{|z|\le ct}\int |z|^{-1}\bar{z}\cdot\nabla_x\phi_2 f^D(\hat{t}(z), x+z, p)\,dp\,dz \\
&\quad +\frac{1}{c^3}\int_{|z|\le ct}\int |z|^{-2}\Big(p^2 - 2(\bar{z}\cdot p)^2\Big) f^D(\hat{t}(z), x+z, p), \\
\phi^D_{t,\mathtt{bd}}(t,x) &= -\frac{1}{c^4 t}\int_{|z|=ct}\int (\bar{z}\cdot p)^2 f_0(0, x+z, p)\,dp\,ds(z).
\end{aligned}$$

5.1.2 Representation of the Nordström force-field

We recall the following representation from [3, Proposition 3; Proposition 4]

$$\phi = \phi_D + \phi_S \tag{5.6}$$

$$\partial_t\phi = \phi_{t,D} + \phi_{t,BD} + \phi_{t,a} + \phi_{t,b} + \phi_{t,c} \tag{5.7}$$

$$\nabla_x\phi = \phi_{x,D} + \phi_{x,BD} + \phi_{x,a} + \phi_{x,b} + \phi_{x,c} \tag{5.8}$$

where

$$\begin{aligned}
\phi_D(t,x) &= \partial_t\left(\frac{t}{4\pi}\int_{|\omega|=1}\phi^0(x+ct\omega)\,d\omega\right) + \frac{t}{4\pi}\int_{|\omega|=1}\phi^1(x+ct\omega)\,d\omega \\
\phi_S(t,x) &= -\frac{1}{c^2}\int_{|z|\le ct}\int |z|^{-1}\gamma(p) f(\hat{t}(z), x+z, p)\,dp\,dz \\
\phi_{t,D}(t,x) &= \partial_t\phi_D(t,x)
\end{aligned}$$

$$
\begin{aligned}
\phi_{t,BD}(t,x) &= -c^{-1}(ct)^{-1}\int_{|z|=ct}\int bd^{\phi_t} f^{\circ}(x+z,p)\,dp\,ds(z) \\
\phi_{t,a}(t,x) &= -c^{-2}\int_{|z|\le ct}\int a^{\phi_t} f(\hat{t}(z),x+z,p)\,dp\,\frac{dz}{|z|^2} \\
\phi_{t,b}(t,x) &= -c^{-2}\int_{|z|\le ct}\int b^{\phi_t} S(\phi) f(\hat{t}(z),x+z,p)\,dp\,\frac{dz}{|z|} \\
\phi_{t,c}(t,x) &= -c^{-1}\int_{|z|\le ct}\int c^{\phi_t}(\nabla_x\phi) f(\hat{t}(z),x+z,p)\,dp\,\frac{dz}{|z|}
\end{aligned}
$$

and

$$
\begin{aligned}
\phi_{x,D}(t,x) &= \nabla_x\phi_D(t,x) \\
\phi_{x,BD}(t,x) &= -c^{-2}(ct)^{-1}\int_{|z|=ct}\int bd^{\phi_x} f^{\circ}(x+z,p)\,dp\,ds(z) \\
\phi_{x,a}(t,x) &= -c^{-2}\int_{|z|\le ct}\int a^{\phi_x} f(\hat{t}(z),x+z,p)\,dp\,\frac{dz}{|z|^2} \\
\phi_{x,b}(t,x) &= -c^{-3}\int_{|z|\le ct}\int b^{\phi_x} S(\phi) f(\hat{t}(z),x+z,p)\,dp\,\frac{dz}{|z|} \\
\phi_{x,c}(t,x) &= -c^{-2}\int_{|z|\le ct}\int c^{\phi_x}(\nabla_x\phi) f(\hat{t}(z),x+z,p)\,dp\,\frac{dz}{|z|}
\end{aligned}
$$

where the kernels are

$$
\begin{aligned}
bd^{\phi_t} &= \gamma(p)(1+c^{-1}\bar{z}\cdot\hat{p})^{-1} \\
a^{\phi_t} &= -\gamma(1+c^{-1}\bar{z}\cdot\hat{p})^{-2}\hat{p}\cdot(\bar{z}+c^{-1}\hat{p}) \\
b^{\phi_t} &= \gamma(1+c^{-1}\bar{z}\cdot\hat{p})^{-2}(\bar{z}+c^{-1}\hat{p})^2 \\
c^{\phi_t} &= \gamma^3(1+c^{-1}\bar{z}\cdot\hat{p})^{-2}(\bar{z}+c^{-1}\hat{p})
\end{aligned}
$$

and

$$
\begin{aligned}
bd^{\phi_x} &= \gamma(p)(1+c^{-1}\bar{z}\cdot\hat{p})^{-1}\bar{z} \\
a^{\phi_x} &= \gamma(1+c^{-1}\bar{z}\cdot\hat{p})^{-2}\big\{\bar{z}+c^{-1}\hat{p}-c^{-2}\hat{p}\wedge(\bar{z}\wedge\hat{p})\big\} \\
b^{\phi_x} &= \bar{z}b^{\phi_t} \\
c^{\phi_x} &= \bar{z}\oplus c^{\phi_t}\in\mathbb{R}^{3\times 3}.
\end{aligned}
$$

Next we expand these kernels in powers of c^{-1} (in this subsection we will explain the argument leading to a formula for $\nabla_x\phi$ in detail and only give the formulas for the other fields). According to (2.6) we can assume that the p-support of $f(t,x,\cdot)$ is uniformly bounded in $x\in\mathbb{R}^3$ and $t\in[0,T]$, say $f(t,x,p)=0$ for $|p|\ge P$. Thus we may suppose that $|p|\le P$ in each of the p-integrals, and hence also $|\hat{p}|=\gamma(p)|p|\le|p|\le P$ uniformly in c. It follows that

$$\hat{p}=(1-1/2c^{-2}p^2)p+\mathcal{O}(c^{-4}).$$

For instance, for the kernel bd^{ϕ_x} of $\phi_{x,BD}$ this yields

$$bd^{\phi_x} = \bar{z}\gamma(p)(1+\frac{\bar{z}\cdot\hat{p}}{c})^{-1}$$

$$
\begin{aligned}
&= \bar z\Big[1 - 1/2c^{-2}p^2 + \mathcal{O}(c^{-4})\Big]\Big[1 - c^{-1}(\bar z\cdot p) + c^{-2}(\bar z\cdot p)^2 + c^{-3}\Big(1/2(\bar z\cdot p)p^2 - (\bar z\cdot p)^3\Big) + \mathcal{O}(c^{-4})\Big]\\
&= \bar z\Big(1 - c^{-1}(\bar z\cdot p) + c^{-2}\big((\bar z\cdot p)^2 - 1/2p^2\big) + c^{-3}\big((\bar z\cdot p)p^2 - (\bar z\cdot p)^3\big)\Big) + \mathcal{O}(c^{-4}).
\end{aligned}
$$

If we choose $R_0 > 0$ such that $f^\circ(x,p) = 0$ for $|x| \geq R_0$, then

$$
\begin{aligned}
-(ct)^{-1}\int_{|z|=ct}\int_{|p|\leq P}\mathcal{O}(c^{-4})\mathbf{1}_{B(0,R_0)}(x+z)\,dp\,ds(z) &= \Big(ct\int_{|\omega|=1}\mathbf{1}_{B(0,R_0)}(x+ct\omega)\,ds(\omega)\Big)\mathcal{O}(c^{-4})\\
&= \mathcal{O}(c^{-4})
\end{aligned}
$$

by [23, Lemma 1], uniformly in $x \in \mathbb{R}^3$, $t \in [0,T]$ and $c \geq 1$. Therefore we arrive at

$$
\begin{aligned}
&\phi_{x,BD}(t,x)\\
&= -c^{-2}(ct)^{-1}\int_{|z|=ct}\int \bar z\Big(1 - c^{-1}(\bar z\cdot p) + c^{-2}\big((\bar z\cdot p)^2 - 1/2p^2\big) + c^{-3}\big((\bar z\cdot p)p^2 - (\bar z\cdot p)^3\big)\Big)\times\\
&\quad\times f^\circ(x+z,p)\,dp\,ds(z) + \mathcal{O}(c^{-6}). \qquad (5.9)
\end{aligned}
$$

Concerning $\phi_{x,a}$, we note that $f(t,x,p) = 0$ for $|x| \geq R_0 + TP =: R_1$. Since, by distinguishing the cases $|x-y| \geq 1$ and $|x-y| \leq 1$,

$$
\int_{|z|\leq ct}|z|^{-2}\,\mathbf{1}_{B(0,R_1)}(x+z)\,dz = \int_{|x-y|\leq ct}|x-y|^{-2}\,\mathbf{1}_{B(0,R_1)}(y)\,dy = \mathcal{O}(1)
$$

uniformly in $x \in \mathbb{R}^3$, $t \in [0,T]$, and $c \geq 1$, similar computations as before show that

$$
\begin{aligned}
\phi_{x,a}(x,t) &= -c^{-2}\int_{|z|\leq ct}\int\Big(\bar z + c^{-1}\big[p - 2(\bar z\cdot p)\bar z\big] + c^{-2}\big[3(\bar z\cdot p)^2\bar z - 3/2p^2\bar z - (\bar z\cdot p)p\big]\\
&\qquad c^{-3}\big[4\bar z(\bar z\cdot p)p^2 - 4(\bar z\cdot p)^3\bar z + (\bar z\cdot p)^2 p - p^2 p\big]\Big)\\
&\qquad f(\hat t(z), x+z, p)\,dp\,\frac{dz}{|z|^2} + \mathcal{O}(c^{-6}). \qquad (5.10)
\end{aligned}
$$

In the same manner, elementary calculations using also (2.7) can be carried out to get

$$
\begin{aligned}
\phi_{x,b}(t,x) &= -c^{-3}\int_{|z|\leq ct}\int \bar z S(\phi) f(\hat t(z), x+z, p)\,dp\,\frac{dz}{|z|} + \mathcal{O}(c^{-4}) \qquad (5.11)\\
\phi_{x,c}(t,x) &= -c^{-2}\int_{|z|\leq ct}\int \bar z\oplus\Big(\bar z + c^{-1}\Big[-2(\bar z\cdot p)\bar z + p\Big]\Big)\\
&\qquad(\nabla_x\phi) f(\hat t(z), x+z, p)\,dp\,\frac{dz}{|z|} + \mathcal{O}(c^{-4}), \qquad (5.12)\\
\phi_{t,BD}(x,t) &= -c^{-1}(ct)^{-1}\int_{|z|=ct}\int\Big(1 - c^{-1}(\bar z\cdot p) + c^{-2}\big((\bar z\cdot p)^2 - 1/2p^2\big)\Big)\\
&\qquad f^\circ(x+z,p)\,dp\,ds(z) + \mathcal{O}(c^{-4}), \qquad (5.13)\\
\phi_{t,a}(x,t) &= c^{-2}\int_{|z|\leq ct}\int\Big((\bar z\cdot p) + c^{-1}\big(p^2 - 2(\bar z\cdot p)\big)\Big) f(\hat t(z), x+z, p)\,dp\,\frac{dz}{|z|^2}\\
&\qquad + \mathcal{O}(c^{-4}), \qquad (5.14)
\end{aligned}
$$

$$\phi_{t,b}(t,x) \;=\; -c^{-2}\int_{|z|\le ct}\int S(\phi)f(\hat{t}(z),x+z,p)\,dp\,\frac{dz}{|z|}+\mathcal{O}(c^{-2}), \tag{5.15}$$

$$\phi_{t,c}(t,x) \;=\; -c^{-1}\int_{|z|\le ct}\int \bar{z}\cdot\nabla_x\phi f(\hat{t}(z),x+z,p)\,dp\,\frac{dz}{|z|}+\mathcal{O}(c^{-2}), \tag{5.16}$$

$$\phi_{S}(t,x) \;=\; -\frac{1}{c^2}\int_{|z|\le ct}\int f(\hat{t}(z),x+z,p)\,dp\,\frac{dz}{|z|}+\mathcal{O}(c^{-4}). \tag{5.17}$$

Next we consider the data terms treating first the gradient term.

$$\begin{aligned}
\phi_{x,D} &= \nabla_x\partial_t\Big(\frac{1}{4\pi}\int_{|\omega|=1}\phi^0(x+ct\omega)\,d\omega)\Big)+\nabla_x\Big(\frac{t}{4\pi}\int_{|\omega=1}\phi^1(x+ct\omega)\,d\omega\Big)\\
&= \partial_t\Big(\frac{1}{4\pi}\int_{|\omega|=1}\nabla_x\phi^0(x+ct\omega)\,d\omega)\Big)+\frac{t}{4\pi}\int_{|\omega=1}\nabla_x\phi^1(x+ct\omega)\,d\omega\\
&= I+II
\end{aligned} \tag{5.18}$$

Since $f_2(0,x,p)=0$ by (1.4), we have $\int f_2(0,x,p)\,dp=0$. Thus we get from (1.9), (VP) and (1.6)

$$\begin{aligned}
\nabla_x\phi^0(x) &= c^{-2}\nabla_x\phi_2(0,x)+c^{-4}\nabla_x\phi_4(0,x)\\
&= -\frac{1}{c^2}\int\int|z|^{-2}\bar{z}(1-\frac{p^2}{2c^2})f_0(0,x+z,p)\,dp\,dz+\frac{1}{2c^4}\int\bar{z}\partial_t^2\mu_0(0,x+z)\,dz.
\end{aligned}$$

Using the formulas (5.32)-(5.34) below, we calculate

$$\begin{aligned}
&\frac{-1}{c^2}\int_{|\omega|=1}\int\int|z|^{-2}\bar{z}\big(1-\frac{p^2}{2c^2}\big)f_0(0,x+ct\omega+z,p)\,dp\,dz\,d\omega\\
&\quad= \frac{-1}{c^2}\int\int(1-\frac{p^2}{2c^2})f_0(0,y,p)\,dp\int_{|\omega|=1}|y-x-ct\omega|^{-3}(y-x-ct\omega)\,d\omega\,dy\\
&\quad= \frac{-4\pi}{c^2}\int_{|z|\ge ct}\int|z|^{-2}\bar{z}(1-\frac{p^2}{2c^2})f_0(0,x+z,p)\,dp\,dz,\\
&\frac{1}{2c^4}\int_{|\omega|=1}\int\bar{z}\partial_t^2\mu_0(0,x+ct\omega+z)\,dz\,d\omega\\
&\quad= \frac{1}{2c^4}\int\partial_t^2\mu_0(0,y)\int_{|\omega|=1}|y-x-ct\omega|^{-1}(y-x-ct\omega)\,d\omega\,dy\\
&\quad= \frac{2\pi}{c^4}\int_{|z|\ge ct}(\bar{z}-\frac{1}{3}(ct)^2|z|^{-2}\bar{z})\partial_t^2\mu_0(0,x+z)\,dz+\frac{4\pi}{3c^5t}\int_{|z|\le ct}z\partial_t^2\mu_0(0,x+z)\,dz.
\end{aligned}$$

Therefore we get

$$\begin{aligned}
I &= \partial_t\Big(\frac{t}{4\pi}\int_{|\omega|=1}\phi^0(x+ct\omega)\,d\omega\Big) \qquad (5.19)\\
&= \partial_t\Big\{\frac{-t}{c^2}\int_{|z|\ge ct}\int|z|^{-2}\bar{z}(1-\frac{p^2}{2c^2})f_0(0,x+z,p)\,dp\,dz\\
&\quad+\frac{t}{2c^4}\int_{|z|\ge ct}\Big(\bar{z}-\frac{(ct)^2}{3}|z|^{-2}\bar{z}\Big)\partial_t^2\mu_0(0,x+z)\,dz+\frac{1}{3c^5}\int_{|z|\le ct}z\partial_t^2\mu_0(0,x+z)\,dz\Big\}\\
&= \frac{-1}{c^2}\int_{|z|\ge ct}\int|z|^{-2}\bar{z}\big(1-\frac{p^2}{2c^2}\big)f_0(0,x+z,p)\,dp\,dz+\frac{1}{2c^4}\int_{|z|\ge ct}\bar{z}\partial_t^2\mu_0(0,x+z)\,dz
\end{aligned}$$

$$-\frac{t^2}{2c^2}\int_{|z|\geq ct}|z|^{-2}\bar z\partial_t\mu_0(0,x+z)\,dz+\frac{1}{c^3t}\int_{|z|=ct}\int\bar z\Big(1-\frac{p^2}{2c^2}\Big)f_0(0,x+z,p)\,dp\,ds(z),$$

note that several terms have canceled here. A similar calculation yields

$$\begin{aligned}II=&-\frac{t}{c^2}\int_{|z|\geq ct}|z|^{-2}\int\bar z\Big(\partial_t f_0+\frac{\partial_t f_2}{c^2}-\frac{p^2\partial_t f_0}{2c^2}\Big)(0,x+z,p)\,dp\,dz\\&+\frac{t}{2c^4}\int_{|z|\geq ct}\bar z\partial_t^3\mu_0(0,x+z)\,dz-\frac{t^3}{6c^2}\int_{|z|\geq ct}|z|^{-2}\bar z\partial_t^3\mu_0(0,x+z)\,dz\\&+\frac{1}{3c^5}\int_{|z|\leq ct}z\partial_t^3\mu_0(0,x+z)\,dz.\end{aligned}\tag{5.20}$$

We need to examine the last term in (5.20) more closely. Employing the Vlasov-equation and integration by parts twice, we have

$$\begin{aligned}\frac{1}{3c^5}\int_{|z|\leq ct}z\partial_t^3\mu_0(0,x+z)\,dz=&-\frac{t}{3c^4}\int_{|z|=ct}\bar z(\bar z\cdot p)\partial_t^2\mu_0(0,x+z)\,ds(z)\\-\frac{1}{3c^5}\int_{|z|=ct}\int(\bar z\cdot p)p\partial_t f_0(0,x+z,p)\,dp\,ds(z)&-\frac{1}{3c^5}\int_{|z|\leq ct}\partial_t\big(\mu_0\nabla_x\phi_2\big)(0,x+z)\,dz.\end{aligned}\tag{5.21}$$

Using the support properties and the bounds (2.1)–(2.4) as well as [23, Lemma 1] for the integrals over the boundary yields

$$\nabla\phi_D=\mathcal{O}(c^{-2}).\tag{5.22}$$

Therefore, (2.6), (2.7), together with (5.9)–(5.12) and (5.22) gives

$$\nabla\phi=\mathcal{O}(c^{-2})\tag{5.23}$$

and the error estimate in (5.12) is improved to

$$\begin{aligned}\phi_{x,c}(t,x)\;=\;&-c^{-2}\int_{|z|\leq ct}\int\bar z\oplus\Big(\bar z+c^{-1}\Big[-2(\bar z\cdot p)\bar z+p\Big]\Big)\\&(\nabla_x\phi)f(\hat t(z),x+z,p)\,dp\,\frac{dz}{|z|}+\mathcal{O}(c^{-6}).\end{aligned}\tag{5.24}$$

We next claim that also

$$\partial_t\phi=\mathcal{O}(c^{-2}),$$

which gives the improved error estimate in (5.11)

$$\phi_{x,b}(t,x)\;=\;-c^{-3}\int_{|z|\leq ct}\int\bar zS(\phi)f(\hat t(z),x+z,p)\,dp\,\frac{dz}{|z|}+\mathcal{O}(c^{-6}).\tag{5.25}$$

Combining (5.8)–(5.10), (5.25), (5.24) and (5.19)–(5.21), we obtain the announced formulas (3.5)–(3.7).

In order to prove the claim and to establish representation formulas we need to examine ϕ_D and $\partial_t\phi_D$ more closely. Although we only want to give an approximation up to order c^{-3}, we have to take into account the terms coming from initial data of order c^{-4}, too, because the associated homogenous fields have contributions of lower order at least in the case of $\partial_t\phi_D$. Therefore we

have to give the full description of ϕ_D and $\partial_t\phi_D$. Since the calculations are similar to that already carried out, we restrict ourself to give the formulas only:

$$\begin{aligned}
\phi_D(t,x) &= -\frac{1}{c^2}\int_{|z|\geq ct}|z|^{-1}\Big(\mu_0+t\partial_t\mu_0\Big)(0,x+z)\,dz+\frac{1}{c^3}\int_{|z|=ct}\int(\bar z\cdot p)f_0(0,x+z,p)\,dp\,dz\\
&+\frac{1}{2c^4}\int_{|z|\geq ct}\int\Big(|z|^{-1}p^2f_0-(\bar z\cdot p)\partial_tf_0\Big)(0,x+z,p)\,dp\,dz\\
&+\frac{t^2}{2c^2}\int_{|z|\geq ct}\int|z|^{-2}(\bar z\cdot p)\partial_tf_0(0,x+z,p)\,dp\,dz\\
&+\frac{1}{c^5}\int_{|z|\leq ct}\int\Big(-\partial_tf_2+\frac{p^2}{2}\partial_tf_0-\frac{|z|}{3}(\bar z\cdot p)\partial_t^2f_0\Big)(0,x+z,p)\,dp\,dz\\
&-\frac{t}{c^4}\int_{|z|\geq ct}\int|z|^{-1}\Big(\partial_tf_2+\frac{p^2}{2}\partial_tf_0\Big)(0,x+z,p)\,dp\,dz\\
&-\frac{t}{2c^4}\int_{|z|\geq ct}\int(\bar z\cdot p)\partial_t^2f_0(0,x+z,p)\,dp\,dz\\
&+\frac{t^3}{6c^2}\int_{|z|\geq ct}\int|z|^{-2}(\bar z\cdot p)\partial_t^2f_0(0,x+z,p)\,dp\,dz. \qquad (5.26)
\end{aligned}$$

Taking into account the by now well known support properties and bounds from (2.1)–(2.4) as well as e.g.

$$\begin{aligned}
\frac{t^2}{c^2}\int_{|z|\geq ct}|z|^{-2}\mathbf{1}_{B(0,M_0)}(x+z)\,dz &\leq \frac{1}{c^4}\int_{|z|\geq ct}\mathbf{1}_{B(0,M_0)}(x+z)\,dz\leq Mc^{-4},\\
\frac{1}{c^5}\int_{|z|\leq ct}|z|\mathbf{1}_{B(0,M_0)}(x+z)\,dz &\leq \frac{t}{c^4}\int_{|z|\leq ct}\mathbf{1}_{B(0,M_0)}(x+z)\,dz\leq Mc^{-4}
\end{aligned}$$

(5.26) gives

$$\phi_D(t,x)=-\frac{1}{c^2}\int_{|z|\geq ct}|z|^{-1}\Big(\mu_0+t\partial_t\mu_0\Big)(0,x+z)\,dz+\frac{1}{c^3}\int_{|z|=ct}\int(\bar z\cdot p)f_0(0,x+z,p)\,dp\,dz+\mathcal{O}(c^{-4}). \qquad (5.27)$$

Combining (5.6), (5.17) and (5.27) we obtain

$$\phi=\phi_{\mathtt{ext}}+\phi_{\mathtt{int}}+\phi_{\mathtt{bd}}+\mathcal{O}(c^{-4}) \qquad (5.28)$$

with

$$\begin{aligned}
\phi_{\mathtt{ext}}(t,x) &= -\frac{1}{c^2}\int_{|z|\geq ct}|z|^{-1}\Big(\mu_0+t\partial_t\mu_0\Big)(0,x+z)\,dz,\\
\phi_{\mathtt{int}}(t,x) &= -\frac{1}{c^2}\int_{|z|\leq ct}\int f(\hat t(z),x+z,p)\,dp\,\frac{dz}{|z|},\\
\phi_{\mathtt{bd}}(t,x) &= \frac{1}{c^3}\int_{|z|=ct}\int(\bar z\cdot p)f_0(0,x+z,p)\,dp\,dz.
\end{aligned}$$

Differentiating (5.26) w.r.t t and applying similar estimates yields

$$\partial_t\phi_D(t,x) = \frac{1}{c^2t}\int_{|z|=ct}\mu_0(0,x+z)\,ds(z)+\frac{1}{c^2}\int_{|z|\geq ct}\int|z|^{-2}(\bar z\cdot p)f_0(0,x+z,p)\,dp\,dz$$

$$-\frac{1}{c^3 t}\int_{|z|=ct}\int(\bar z\cdot p)f_0(0,x+z,p)\,dp\,ds(z)-\frac{1}{2c^4 t}\int_{|z|=ct}\int p^2 f_0(0,x+z,p)\,dp\,ds(z)$$
$$+\frac{t}{c^2}\int_{|z|\geq ct}\int |z|^{-2}(\bar z\cdot p)\partial_t f_0(0,x+z,p)\,dp\,dz+\mathcal{O}(c^{-4}). \tag{5.29}$$

Combining (5.7), (5.13)–(5.16) and (5.29) we have

$$\partial_t\phi=\phi_{t,\mathtt{ext}}+\phi_{t,\mathtt{int}}+\phi_{t,\mathtt{bd}}+\mathcal{O}(c^{-4}) \tag{5.30}$$

with

$$\begin{aligned}
\phi_{t,\mathtt{ext}}(t,x) &= \frac{1}{c^2}\int_{|z|\geq ct}\int |z|^{-2}(\bar z\cdot p)(f_0+t\partial_t f_0)(0,x+z,p)\,dp\,dz,\\
\phi_{t,\mathtt{int}}(t,x) &= \frac{1}{c^2}\int_{|z|\leq ct}\int \Big((\bar z\cdot p)+c^{-1}\big(p^2-2(\bar z\cdot p)\big)\Big)f(\hat t(z),x+z,p)\,dp\,\frac{dz}{|z|^2}\\
&\quad -\frac{1}{c}\int_{|z|\leq ct}\int \bar z\cdot\nabla_x\phi f(\hat t(z),x+z,p)\,dp\,\frac{dz}{|z|},\\
\phi_{t,\mathtt{bd}}(t,x) &= -\frac{1}{c^4 t}\int_{|z|=ct}\int(\bar z\cdot p)^2 f_0(0,x+z,p)\,dp\,ds(z).
\end{aligned}$$

Note, that because of (5.22) and (5.13)–(5.16) and (5.29) we already have

$$\partial_t\phi=\mathcal{O}(c^{-2}), \tag{5.31}$$

which in turn gives

$$\begin{aligned}
\phi_{t,b}(t,x) &= \mathcal{O}(c^{-4})\\
\phi_{t,c}(t,x) &= -c^{-1}\int_{|z|\leq ct}\int \bar z\cdot\nabla_x\phi f(\hat t(z),x+z,p)\,dp\,\frac{dz}{|z|}+\mathcal{O}(c^{-4}),
\end{aligned}$$

compare with (5.15) and (5.16).

5.2 Some explicit integrals

We point out some formulas that have been used in the previous sections. For $z\in\mathbb{R}^3$ and $r>0$ an elementary calculation yields

$$\int_{|\omega|=1}|z-r\omega|^{-1}\,d\omega=\left\{\begin{array}{ccc}4\pi r^{-1} & : & r\geq|z|\\ 4\pi|z|^{-1} & : & r\leq|z|\end{array}\right. . \tag{5.32}$$

Differentiation w.r.t. z gives

$$\int_{|\omega|=1}|z-r\omega|^{-3}(z-r\omega)\,d\omega=\left\{\begin{array}{ccc}0 & : & r>|z|\\ 4\pi|z|^{-2}\bar z & : & r<|z|\end{array}\right. . \tag{5.33}$$

Similarly,

$$\int_{|\omega|=1}|z-r\omega|\,d\omega=\left\{\begin{array}{ccc}4\pi r+\frac{4\pi}{3}z^2r^{-1} & : & r\geq|z|\\ 4\pi|z|+\frac{4\pi}{3}r^2|z|^{-1} & : & r\leq|z|\end{array}\right. ,$$

and thus by differentiation

$$\int_{|\omega|=1} |z-r\omega|^{-1}(z-r\omega)\,d\omega = \left\{ \begin{array}{ccc} \frac{8\pi}{3r}\,z & : & r>|z| \\ 4\pi\bar{z} - \frac{4\pi}{3}r^2|z|^{-2}\bar{z} & : & r<|z| \end{array} \right. . \tag{5.34}$$

Finally, for $z \in \mathbb{R}^3 \setminus \{0\}$ also

$$\int |z-v|^{-1}|v|^{-3}v\,dv = 2\pi\bar{z} \tag{5.35}$$

can be computed.

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